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On Baxter Q-operators for the Toda chain

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Abstract. We suggest a procedure for the construction of Baxter Q-operators for the Toda chain. Apart from the one-parametric family of Q-operators, considered in our recent paper (Pronko 2000 *Commun. Math. Phys.* to appear) we also give the method of construction of two basic Q-operators and the derivation of the functional relations for these operators. Also we have found the relation of the basic Q-operators with Bloch solutions of the quantum linear problem.

1. Introduction

Long ago, in his famous papers [1], Baxter introduced the object that is now known as the Q-operator. This operator was used initially for the solution of the eigenvalue problem of the *XYZ*-spin chain, where the usual Bethe ansatz fails. Recently, this operator was discussed intensively in a series of papers [2] in the connection with continuous quantum field theory. In [3] the relation between Q-operator and quantum Bäklund transformations was pointed out. In [4] we suggested the construction of a one-parametric family of Q-operators for the most difficult case of the isotropic Heisenberg spin chain. (In spite of the obvious simplicity of this model, the original Baxter construction fails here.)

The existence of the one-parametric family of Q-operators implies the existence of two basic solutions to the Baxter equation, whose linear combinations (with operator coefficients) form the one-parametric family.

In the present paper we extend the investigation started in [4] to the periodic Toda chain, the other model with a rational R-matrix. It turns out that apart from the construction of the one-parametric family of Q-operators (section 2), in the case of the Toda chain it is possible to also build two basic Q-operators separately (section 3). These basic operators satisfy the set of functional Wronskian relations (section 5), first established for a certain field-theoretical model in [2]. On the one hand, the Wronskian relations imply the linear independence of the basic operators, on the other hand, they are the origin for numerous fusion relations for the transfer matrix of the model.

In our approach we construct the basic Q-operators as the trace of the monodromy of certain $M_n^{(1,2)}(x)$ operators (section 3). It turns out that these operators also permit us to construct the quantum Bloch functions, the basis of the solutions of the quantum linear problem, which are the eigenvectors of the monodromy matrix (section 6).

The defining relation of the Q-operator (Baxter equation) for the models with rational R-matrix looks as follows:

$$t(x)Q(x) = a(x)Q(x+\mathrm{i}) + b(x)Q(x-\mathrm{i}) \tag{1}$$

where $t(x)$ is the corresponding transfer matrix and $a(x)$ and $b(x)$ are the c-number functions which enter into factorization of the quantum determinant of $t(x)$. In the case of the Toda chain the quantum determinant is unity, therefore we can choose the normalization $a(x) = b(x) = 1$, which we shall use below.

2. Toda chain

The periodic Toda chain is the quantum system described by the Hamiltonian

$$H = \sum_{i=1}^{N} \left(p_i^2/2 + \exp(q_{i+1} - q_i)\right) \tag{2}$$

where the canonical variables p_i, q_i satisfy commutation relations

$$[p_i, q_j] = \mathrm{i}\delta_{ij} \tag{3}$$

and periodic boundary conditions

$$\begin{aligned} p_{i+N} &= p_i \\ q_{i+N} &= q_i. \end{aligned} \tag{4}$$

Following Sklyanin [5] we introduce the Lax operator in two-dimensional auxiliary space as follows:

$$L_n(x) = \begin{pmatrix} x - p_n & \mathrm{e}^{q_n} \\ -\mathrm{e}^{-q_n} & 0 \end{pmatrix} \tag{5}$$

where x is the spectral parameter. The fundamental commutation relations for the Lax operator could be written in R-matrix form:

$$R_{12}(x-y)L_n^1(x)L_n^2(y) = L_n^2(y)L_n^1(x)R_{12}(x-y) \tag{6}$$

where the indices 1, 2 indicate different auxiliary spaces and the R-matrix is given by

$$R_{12}(x) = x + \mathrm{i}P_{12} \tag{7}$$

where P is the operator of permutation of the auxiliary spaces. The same intertwining relation also holds true and for the monodromy matrix corresponding to the L-operator (5):

$$T_{ij}(x) = \left(\prod_1^N L_n(x)\right)_{ij} \tag{8}$$

where the multipliers of the product are ordered from right to left.

The $Q(x)$-operator we are going to construct will be given as the trace of the monodromy $\hat{Q}(x)$ of appropriate operators $M_n(x)$, which acts in the nth quantum space and its auxiliary space, which we will choose to be the representation space Γ of the algebra

$$[\rho_i, \rho_j^+] = \delta_{ij} \qquad i, j = 1, 2. \tag{9}$$

The operator $\hat{Q}(x)$ will be given by the ordered product

$$\hat{Q}(x) = \prod_{n=1}^{N} M_n(x). \tag{10}$$

Furthermore, we shall need to consider the product $(L_n(x))_{ij}M_n(x)$, which acts in the auxiliary space $\Gamma \times C^2$ (Γ for $M_n(x)$ and C^2 is the two-dimensional auxiliary space for $L_n(x)$). In this space it is convenient to consider a pair of projectors $\Pi^{\pm}_{ij}$:

$$\begin{aligned} \Pi^{+}_{ij} &= (\rho^{+}\rho + 1)^{-1}\rho_i\rho^{+}_j = \rho_i\rho^{+}_j(\rho^{+}\rho + 1)^{-1} \\ \Pi^{-}_{ij} &= (\rho^{+}\rho + 1)^{-1}\epsilon_{il}\rho^{+}_l\epsilon_{jm}\rho_m = \epsilon_{il}\rho^{+}_l\epsilon_{jm}\rho_m(\rho^{+}\rho + 1)^{-1} \end{aligned} \tag{11}$$

where

$$\begin{aligned} &\rho^{+}\rho = \rho^{+}_i\rho_i \\ &\epsilon_{ij} = -\epsilon_{ji} \qquad \epsilon_{12} = 1. \end{aligned} \tag{12}$$

These projectors formally satisfy the following relations:

$$\begin{aligned} &\Pi^{\pm}_{ik}\Pi^{\pm}_{kj} = \Pi^{\pm}_{ij} \\ &\Pi^{+}_{ik}\Pi^{-}_{kj} = 0 \\ &\Pi^{+}_{ij} + \Pi^{-}_{ij} = \delta_{ij}. \end{aligned} \tag{13}$$

Rigorously speaking, the right-hand side of the first equation (13) in the Fock representation has an extra term, proportional to the projector on the vacuum state, but, as we shall see below, this term is irrelevant in the present discussion.

In order to define the Q-operator which satisfies the Baxter equation we shall exploit Baxter's idea [1], which we reformulate as follows: the *$M_n(x)$-operator should satisfies the relation*

$$\Pi^{-}_{ij}(L_n(x))_{jl}M_n(x)\Pi^{+}_{lk} = 0. \tag{14}$$

If this condition is fulfilled, then

$$\begin{aligned} (L_n(x))_{ij}M_n(x) = {} & \Pi^{+}_{ik}\,(M_n(x))_{kl}\,M_n(x)\Pi^{+}_{lj} \\ & +\Pi^{-}_{ik}\,(L(x)_n)_{kl}\,M_n(x)\Pi^{-}_{lj} + \Pi^{+}_{ik}(L_n(x))_{kl}M_n(x)\Pi^{-}_{lj}. \end{aligned} \tag{15}$$

In other words, condition (14) guarantees that the right-hand side of (15), in the sense of projectors $\Pi^{\pm}$ has the triangle form and this form will be conserved for products over n due to orthogonality of the projectors.

From (14) we obtain

$$\epsilon_{jm}\rho_m(L_n(x))_{jk}M_n(x)\rho_k = 0. \tag{16}$$

To satisfy this equation it is sufficient if

$$M_n(x)\rho_k = (L^{-1}_n(x))_{kl}\rho_l A_n(x) \tag{17}$$

or

$$\epsilon_{jm}\rho_m(L_n(x))_{jk}M_n(x) = B_n(x)\epsilon_{kl}\rho_l \tag{18}$$

where $A_n(x)$ and $B_n(x)$ are some operators which we shall now find. Note that the operator $L^{-1}_n(x)$ is given by

$$L^{-1}_n(x) = \begin{pmatrix} 0 & -\mathrm{e}^{q_n} \\ \mathrm{e}^{-q_n} & x - i - p_n \end{pmatrix} \tag{19}$$

Equation (18) could be rewritten in the following form:

$$\left(L^{-1}_n(x + \mathrm{i})\right)_{jk}\rho_k M_n(x) = B_n(x)\rho_j. \tag{20}$$

Comparing equations (17) and (20) we conclude that they both are satisfied provided

$$\begin{aligned} A_n(x) &= M_n(x-\mathrm{i}) \\ B_n(x) &= M_n(x+\mathrm{i}). \end{aligned} \tag{21}$$

In such a way we obtain the following equation for the $M(x)$-operator:

$$\left(L_n^{-1}(x+\mathrm{i})\right)_{jk} \rho_k M_n(x) = M_n(x+\mathrm{i})\rho_j. \tag{22}$$

If the operator $M_n(x)$ satisfies this equation, the product $L_n(x)M_n(x)$ takes the following form:

$$\begin{aligned} (L_n(x))_{ij} M_n(x) = {} & \rho_i M_n(x-\mathrm{i})\rho_j^+(\rho^+\rho+1)^{-1} \\ & +(\rho^+\rho+1)^{-1}\epsilon_{il}\rho_l^+ M_n(x+\mathrm{i})\epsilon_{jm}\rho_m + \Pi_{ik}^+(L_n(x))_{kl}M_n(x)\Pi_{lj}^-. \end{aligned} \tag{23}$$

We do not detail the last term in (23) because, due to the triangle structure of its right-hand side this term will not enter into the trace of $\hat{Q}(x)$.

Now our task is to solve the equation for the $M_n(x)$-operator. A detailed investigation of equation (22) shows that the usual Fock representation for (9) is not fit for our purpose, therefore we shall use a less restrictive holomorphic representation.

Let the operator ρ_i^+ be the operator of multiplication by α_i, while the operator ρ_i is the operator of differentiation with respect to α_i:

$$\begin{aligned} \rho_i^+\psi(\alpha) &= \alpha_i\psi(\alpha) \\ \rho_i\psi(\alpha) &= \frac{\partial}{\partial\alpha}\psi(\alpha). \end{aligned} \tag{24}$$

The operators ρ_i^+ and ρ_i are canonically conjugated for the scalar product

$$(\psi,\phi) = \int \frac{\prod_{i=1,2}\mathrm{d}\alpha_i\,\mathrm{d}\bar{\alpha}_i}{(2\pi\mathrm{i})^2}\mathrm{e}^{-\alpha\bar{\alpha}}\bar{\psi}(\alpha)\phi(\alpha). \tag{25}$$

The action of an operator in the holomorphic representation is defined by its kernel:

$$(K\psi)(\alpha) = \int \mathrm{d}^2\mu(\beta)K(\alpha,\bar{\beta})\psi(\beta) \tag{26}$$

where we have denoted

$$\mathrm{d}^2\mu(\beta) = \frac{\prod_{i=1,2}\mathrm{d}\beta_i\,\mathrm{d}\bar{\beta}_i}{(2\pi\mathrm{i})^2}. \tag{27}$$

Now we are ready to make the following statement.

Statement. The kernel $M_n(x,\alpha,\bar{\beta})$ of the operator $M_n(x)$ in holomorphic representation has the following form:

$$M_n(x,\alpha,\bar{\beta}) = m_n(x)\frac{(\alpha\bar{\beta})^{2l+\mathrm{i}x}}{\Gamma(2l+\mathrm{i}x+1)} \tag{28}$$

where l is an arbitrary parameter and the operator $m_n(x)$ is given by

$$\begin{aligned} m_n(x) &= \exp\left[\pi/2(\rho_1^+\rho_2\mathrm{e}^{q_n} - \rho_2^+\rho_1\mathrm{e}^{-q_n}\right]\left(1+\mathrm{i}\rho_2^+\rho_1\mathrm{e}^{-q_n}\right)^{\mathrm{i}(p_n-x)+\rho_1^+\rho_1} \\ &= \left(1-\mathrm{i}\rho_1^+\rho_2\mathrm{e}^{q_n}\right)^{\mathrm{i}(p_n-x)+\rho_1^+\rho_1}\exp\left[\pi/2(\rho_1^+\rho_2\mathrm{e}^{q_n} - \rho_2^+\rho_1\mathrm{e}^{-q_n}\right]. \end{aligned} \tag{29}$$

In (28) the operator $m_n(x)$ acts on the argument α of the function $(\alpha\bar{\beta})^{2l+\mathrm{i}x}$ according to (24). The proof of the *statement* is straightforward by a direct substitution of (28) into equation (22).

This calculation also gives us as a by-product the meaning of the operator $m_n(x)$. Apparently, this operator commutes with the operator

$$\hat{l} = \frac{1}{2}(\rho_1^+\rho_1 + \rho_2^+\rho_2). \tag{30}$$

If we fix the subspace of Γ corresponding to the definite eigenvalue l of the operator $\hat{l}$, then the operator $m_n(x - \mathrm{i}(l + \frac{1}{2}))$ becomes the Lax operator of the Toda chain with auxiliary space, corresponding to spin l. In particular, the operator (5) corresponds to $l = \frac{1}{2}$. Generally speaking, $m_n(x - \mathrm{i}(l + \frac{1}{2}))$ represents the Lax operator of the Toda chain in the auxiliary space Γ. This statement could be proved by intertwining of the operator (5) with $m_n(x - \mathrm{i}(l + \frac{1}{2}))$.

Now, taking the ordered product of the $M_n(x)$ operators we shall obtain the operator $\hat{Q}(x, l)$ whose kernel is given by

$$\begin{aligned}\hat{Q}(x, l, \alpha, \bar{\beta}) = \int \prod_{i=1}^{N-1} \mathrm{d}^2\mu(\gamma_i) M_N(x, l, \alpha, \bar{\gamma}_{N-1}) M_{N-1}(x, l, \gamma_{N-1}, \bar{\gamma}_{N-2}) \\ \times M_2(x, l, \gamma_2, \bar{\gamma}_1) M_1(x, l, \gamma_1, \bar{\beta}).\end{aligned} \tag{31}$$

Due to the triangle (in the sense of projectors $\Pi^\pm$) structure of the right-hand side of (23) we obtain the following rule of multiplication of the monodromy matrix $T(x)$ on the operator $\hat{Q}(x)$:

$$\begin{aligned}(T(x))_{ij}\, \hat{Q}(x, l, \alpha, \bar{\beta}) = (x + \tfrac{1}{2}\mathrm{i})^N \rho_i \hat{Q}(x - \mathrm{i}, l, \alpha, \bar{\beta}) \rho_j^+ (\rho^+\rho + 1)^{-1} \\ \times (x - \tfrac{1}{2}\mathrm{i})^N (\rho^+\rho + 1)^{-1} \epsilon_{im} \rho_m^+ \hat{Q}(x + \mathrm{i}, l, \alpha, \bar{\beta}) \epsilon_{jk} \rho_k + \Pi_{im}^+ (\cdots)_{mk} \Pi_{kj}^-\end{aligned} \tag{32}$$

where we omitted the explicit expression of the last term for an obvious reason.

To proceed further we need to remind the reader of the definition of the trace of an operator in the holomorphic representation. If the operator is given by its kernel $F(\alpha, \bar{\beta})$ then (see, e.g., [6])

$$\operatorname{Tr} F = \int \mathrm{d}^2\mu(\alpha) F(\alpha, \bar{\alpha}) \tag{33}$$

where the measure was defined in (27). Now we can perform the trace operation for both sides of (32) over the holomorphic variables and over i, j indices, corresponding to the auxiliary two-dimensional space of $T(x)$. The result is the desired Baxter equation:

$$t(x) Q(x, l) = Q(x - \mathrm{i}, l) + Q(x + \mathrm{i}, l) \tag{34}$$

where, according to (33)

$$Q(x, l) = \int \mathrm{d}^2\mu(\alpha) \hat{Q}(x, l, \alpha, \bar{\alpha}). \tag{35}$$

Note, that the trace of $\hat{Q}$ exists due to the exponential factor in the holomorphic measure (27) and has a cyclic property, therefore $Q(x, l)$ is invariant under cyclic permutation of the quantum variables. Acting as above we can also consider right multiplication $M_n(x)L_n(x)$ to obtain

$$Q(x, l) t(x) = Q(x - \mathrm{i}, l) + Q(x + \mathrm{i}, l). \tag{36}$$

We shall not consider here the derivations of the intertwining relations for $\hat{Q}(x, l)$ for different values of x and l and for $\hat{Q}(x, l)$ and $T_{ij}(y)$. This may be done in the same way as in [4] and these relations imply the following commutation relations:

$$\begin{aligned}&[Q(x, l), Q(y, m)] = 0 \\ &[t(x), Q(y, l)] = 0.\end{aligned} \tag{37}$$

In such a way we have constructed the family of solutions of the Baxter equation which are parametrized by the parameter l. We can prove that this family may be considered as a linear combinations of two basic solutions with operator coefficients. Here the following question arises. Is it possible to construct these basic operators separately. The answer is positive and now we shall show how our procedure should be modified in this case.

3. Basic Q-operators for the Toda chain

As above, we shall look for the Q-operators in the form of the monodromy of appropriate $M_n^{(i)}(x)$-operators, which we now supply with the index $i = 1, 2$ and which act in nth quantum space. The auxiliary space Γ will now be the representation space of one Heisenberg algebra, instead of (9):

$$[\rho, \rho^+] = 1. \tag{38}$$

The product $(L_n(x))_{ij} M_n^{(i)}(x)$ is an operator in nth quantum space and in auxiliary space which is the tensor product $\Gamma \times C^2$. In this auxiliary space we shall introduce new projectors:

$$\begin{aligned} \Pi_{ij}^+ &= \begin{pmatrix} 1 \\ \rho \end{pmatrix} \frac{1}{\rho^+\rho + 1}(1, \rho^+) \\ \Pi_{ij}^- &= \begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix} \frac{1}{\rho^+\rho + 2}(-\rho, 1) \end{aligned} \tag{39}$$

The defining equations for the operators $M_n^{(i)}$ (the analogues of equation (14)) are

$$\begin{aligned} \Pi_{ik}^-(L_n(x))_{kl} M_n^{(1)}(x) \Pi_{lj}^+ &= 0 \\ \Pi_{ik}^+(L_n(x))_{kl} M_n^{(2)}(x) \Pi_{lj}^- &= 0. \end{aligned} \tag{40}$$

The solutions of these equations we again will present as the kernels of the corresponding operators in holomorphic representation of the algebra (38):

$$\begin{aligned} M_n^{(1)}(x, \alpha, \bar\beta) &= \exp(-\mathrm{i}\bar\beta \mathrm{e}^{q_n}) \frac{\mathrm{e}^{-\pi x/2}}{\Gamma(-\mathrm{i}(x - p_n) + 1)} \exp(\mathrm{i}\alpha \mathrm{e}^{-q_n}) \\ M_n^{(2)}(x, \alpha, \bar\beta) &= \exp(-\mathrm{i}\alpha \mathrm{e}^{-q_n}) \mathrm{e}^{-\pi x/2} \mathrm{e}^{(x - p_n)} \Gamma(-\mathrm{i}(x - p_n)) \exp(\mathrm{i}\bar\beta \mathrm{e}^{q_n}). \end{aligned} \tag{41}$$

For right multiplication by $L_n(x)$ these operators automatically satisfy the following equations:

$$\begin{aligned} \Pi_{ik}^+ M_n^{(1)}(x)(L_n(x))_{kl} \Pi_{lj}^- &= 0 \\ \Pi_{ik}^- M_n^{(2)}(x)(L_n(x))_{kl} \Pi_{lj}^+ &= 0. \end{aligned} \tag{42}$$

The full multiplication rules for the operators $M_n^i(x)$ and $L_n(x)$ have the following form for left multiplication:

$$\begin{aligned} (L_n(x))_{ij} M_n^{(1)}(x) &= \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i M_n^{(1)}(x - \mathrm{i}) \frac{1}{\rho^+\rho + 1}(1, \rho^+)_j \\ &\quad + \begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_i \frac{1}{\rho^+\rho + 2} M_n^{(1)}(x + \mathrm{i})(-\rho, 1)_j + \Pi_{ik}^+(L_n(x))_{kl} M_n^{(1)}(x) \Pi_{lj}^- \\ (L_n(x))_{ij} M_n^{(2)}(x) &= \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i \frac{1}{\rho^+\rho + 1} M_n^{(2)}(x + \mathrm{i})(1, \rho^+)_j \\ &\quad + \begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_i M_n^{(2)}(x - \mathrm{i}) \frac{1}{\rho^+\rho + 2}(-\rho, 1)_j + \Pi_{ik}^- (L_n(x))_{kl} M_n^{(2)}(x) \Pi_{lj}^+ \end{aligned} \tag{43}$$

and for right multiplication:

$$M_n^{(1)}(x)(L_n(x))_{ij} = \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i \frac{1}{\rho^+\rho+1} M_n^{(1)}(x-\mathrm{i})(1, \rho^+)_j$$
$$+ \begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_i M_n^{(1)}(x+\mathrm{i}) \frac{1}{\rho^+\rho+2}(-\rho, 1)_j + \Pi^-_{ik}(L_n(x))_{kl} M_n^{(1)}(x)\Pi^+_{lj} \tag{44}$$

$$M_n^{(2)}(x)(L_n(x))_{ij} = \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i M_n^{(2)}(x+\mathrm{i}) \frac{1}{\rho^+\rho+1}(1, \rho^+)_j$$
$$+ \begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_i \frac{1}{\rho^+\rho+2} M_n^{(2)}(x-\mathrm{i})(-\rho, 1)_j + \Pi^+_{ik}(L_n(x))_{kl} M_n^{(2)}(x)\Pi^-_{lj}. \tag{45}$$

These relations guarantee that the traces of the monodromies, corresponding to both operators $M_n^{(i)}(x)$ satisfy the Baxter equations:

$$\begin{aligned} t(x)Q^{(i)}(x) &= Q^{(i)}(x+\mathrm{i}) + Q^{(i)}(x-\mathrm{i}) \\ Q^{(i)}(x)t(x) &= Q^{(i)}(x+\mathrm{i}) + Q^{(i)}(x-\mathrm{i}). \end{aligned} \tag{46}$$

We shall conclude this section with the calculation of the operators $Q^i(x)$ for the simplest case of one quantum degree of freedom. In this case from (33) we easily obtain

$$Q^{(1)}(x) = \int \frac{\mathrm{d}\alpha\, \mathrm{d}\bar{\alpha}}{2\pi\mathrm{i}} \mathrm{e}^{-\alpha\bar{\alpha}} M^1(x, \alpha, \bar{\alpha}) = \sum_{n=0} \frac{\mathrm{e}^{-\pi x/2}}{n!} \mathrm{e}^{-qn} \frac{1}{\Gamma(-\mathrm{i}(x-p)+1)} \mathrm{e}^{qn}$$
$$= \sum_{n=0} \frac{\mathrm{e}^{-\pi x/2}}{n!\,\Gamma(-\mathrm{i}(x-p)+n+1)} = \mathrm{e}^{-\pi x/2} I_{-\mathrm{i}(x-p)}(2) \tag{47}$$

where $I_\nu(x)$ is the modified Bessel function. The analogous calculations for the second Q-operator gives

$$Q^{(2)}(x) = -\mathrm{e}^{-\pi x/2} \frac{\pi \mathrm{e}^{\pi(x-p)}}{\sin \pi \mathrm{i}(x-p)} \sum_{n=0} \frac{1}{n!\Gamma(\mathrm{i}(x-p)+n+1)}$$
$$= -\mathrm{e}^{-\pi x/2} \frac{\pi \mathrm{e}^{\pi(x-p)}}{\sin \pi \mathrm{i}(x-p)} I_{\mathrm{i}(x-p)}(2). \tag{48}$$

These two expressions could be compared with the results of [7].

4. Intertwining relations

In this section we shall consider the set of intertwining relations among the $L_n(x)$ and $M_n^{(i)}(x)$ operators, which will imply the mutual commutativity of the transfer matrix and $Q^{(i)}(x)$. Let us start with the simplest relation

$$R^{(i)}_{kl}(x-y)(L_n(x))_{lm} M_n^{(i)}(y) = M_n^{(i)}(y)(L_n(x))_{kl} R^{(i)}_{lm}(x-y). \tag{49}$$

From equation (40) it follows that for $x = y$ the $R^{(i)}$-matrices become the corresponding projectors, Π^- for $i = 1$ and Π^+ for $i = 2$. Making use of these properties we easily obtain:

$$R^{(1)}_{kl}(x-y) = \begin{pmatrix} x-y+\mathrm{i}\rho^+\rho & -\mathrm{i}\rho^+ \\ -\mathrm{i}\rho & \mathrm{i} \end{pmatrix} \tag{50}$$

$$R^{(2)}_{kl}(x-y) = \begin{pmatrix} \mathrm{i} & \mathrm{i}\rho^+ \\ \mathrm{i}\rho & x-y+\mathrm{i}+\mathrm{i}\rho^+\rho \end{pmatrix}. \tag{51}$$

The next relation that we shall consider is

$$M_n^{(1)}(x,\rho)M_n^{(2)}(y,\tau)R^{12}(x-y)=R^{12}(x-y)M_n^{(2)}(y,\tau)M_n^{(1)}(x,\rho) \tag{52}$$

where both M-operators act in different auxiliary spaces $\Gamma^{(i)}$ and the mutual quantum space. The R-matrix acts in the tensor product of auxiliary spaces $\Gamma^{(1)}\times\Gamma^{(2)}$. In (52) we have denoted the operators which act in the auxiliary space $\Gamma^{(1)}$ as ρ, ρ^+ and operators in $\Gamma^{(2)}$ as τ, τ^+. From explicit expressions for the M-operators (41) it follows that

$$\begin{aligned}&(\rho+\tau)M_n^{(1)}(x,\rho)M_n^{(2)}(y,\tau)=0\\&M_n^{(2)}(y,\tau)M_n^{(1)}(x,\rho)(\rho^++\tau^+)=0.\end{aligned} \tag{53}$$

These relations mean that the products of the M-operators are triangle operators in $\Gamma^{(1)}\times\Gamma^{(2)}$ and, as a result the R-matrix satisfies the following equations:

$$\begin{aligned}&(\rho+\tau)R^{12}(x)=0\\&R^{12}(x)(\rho^++\tau^+)=0.\end{aligned} \tag{54}$$

The corollary of (54) is that the kernel of the R-matrix in the holomorphic representation depends on only one variable:

$$R^{12}(x,\alpha,\bar\beta;\gamma,\bar\delta)=f(x,(\alpha-\gamma)(\bar\beta-\bar\delta)) \tag{55}$$

where the variables $\alpha,\bar\beta$ refer to the operators ρ,ρ^+ and variables $\gamma,\bar\delta$ to the operators τ,τ^+. Taking (55) into account we can write the intertwining relation (52) in a holomorphic representation:

$$\begin{aligned}&\int d\mu(\beta')\,d\mu(\delta')\,M_n^{(1)}(x,\alpha,\bar\beta')M_n^{(2)}(y,\gamma,\bar\delta')f(x-y,(\beta'-\delta')(\bar\beta-\bar\delta))\\&\quad=\int d\mu(\alpha')\,d\mu(\gamma')\,f(x-y,(\alpha-\gamma)(\bar\alpha'-\bar\gamma'))M_n^{(2)}(y,\gamma',\bar\delta)M_n^{(1)}(x,\alpha',\bar\beta)\end{aligned} \tag{56}$$

where

$$d\mu(\alpha)=\frac{d\alpha\,d\bar\alpha}{2\pi i}e^{-\alpha\bar\alpha}. \tag{57}$$

To simplify this equation let us introduce the new external variables:

$$\begin{aligned}\xi_1&=\frac{1}{\sqrt2}(\alpha+\gamma)\qquad&\xi_1'&=\frac{1}{\sqrt2}(\beta+\delta)\\\xi_2&=\frac{1}{\sqrt2}(\alpha-\gamma)\qquad&\xi_2'&=\frac{1}{\sqrt2}(\beta-\delta)\end{aligned} \tag{58}$$

and new integration variables for left-hand side (right-hand side) integral:

$$\begin{aligned}\xi_1''&=\frac{1}{\sqrt2}(\beta'+\delta')\qquad&\left(\xi_1''=\frac{1}{\sqrt2}(\alpha'+\gamma')\right)\\\xi_2''&=\frac{1}{\sqrt2}(\beta'-\delta')\qquad&\left(\xi_2''=\frac{1}{\sqrt2}(\alpha'-\gamma')\right).\end{aligned} \tag{59}$$

Apparently, due to the structure of $M^{(i)}$-operators and R-matrix, both sides of (56) depend only on the variables $\xi_2,\bar\xi_2'$ and integration over ξ_1'' becomes trivial, resulting in elimination of these variables in the integrands. Furthermore, representing the function $f(x,2\xi''\bar\xi')$ as

$$f(x,2\xi''\bar\xi')=\sum_{n=0}C_n(x)\frac{(2\xi''\bar\xi')^n}{n!} \tag{60}$$

we can perform the integration over ξ_2'' and, comparing similar terms in both sides of (56), conclude that

$$C_n(x) = \frac{1}{\Gamma(-\mathrm{i}x + n + 1)}. \tag{61}$$

Therefore, the R-matrix in (52) has the following form in the holomorphic representation:

$$R^{12}(x, \alpha, \bar{\beta}; \gamma, \bar{\delta}) = \sum_{n=0} \frac{\left((\alpha - \gamma)(\bar{\beta} - \bar{\delta})\right)^n}{n!\Gamma(-\mathrm{i}x + n + 1)}. \tag{62}$$

As the operator in the space $\Gamma^{(1)} \times \Gamma^{(2)}$, the R-matrix (62) is pathological because its kernel depends only on part of the holomorphic variables. In other words, it contains the projector π on the subspace of $\Gamma^{(1)} \times \Gamma^{(2)}$ which is formed by the functions depending on the difference of variables. This property may be an obstacle in the derivation of the commutativity of Q-operators from the intertwining relation (52). The situation is saved due to the same pathological nature of the product of the M-operators. Indeed, let us consider the product

$$Q^{(1)}(x)Q^{(2)}(y) = \mathrm{Tr}_1 \prod_{k=1}^{N} M_k^{(1)}(x)\, \mathrm{Tr}_2 \prod_{k=1}^{N} M_k^{(2)}(y) = \mathrm{Tr}_{1,2} \prod_{k=1}^{N} M_k^{(1)}(x) M_k^{(2)}(y) \tag{63}$$

where the indices 1, 2 mark the corresponding auxiliary space. Due to the property (53) we can supply each term $M_k^{(1)}(x)M_k^{(2)}(y)$ in the last product with the projector π. The same also holds true for the product of Q-operators taken in the inverse order. In such a way for the commutativity of Q-operators we need to consider only the intertwining relations of M-operators projected onto the space $\pi\left(\Gamma^{(1)} \times \Gamma^{(2)}\right)$, where our R-matrix is well defined.

Next we shall consider the intertwining relation for the $M^{(1)}$-operators with different values of the spectral parameter:

$$R^{(11)}(x - y)M^{(1)}(x, \rho)M^{(1)}(y, \tau) = M^{(1)}(y, \tau)M^{(1)}(x, \rho)R^{(11)}(x - y). \tag{64}$$

As above, the R-matrix in (64) acts in the space $\Gamma^{(1)} \times \Gamma^{(2)}$. From an explicit expression for $M^{(1)}$-operator (41) we obtain

$$\rho M^{(1)}(x, \rho) = M^{(1)}(x, \rho)\mathrm{i}\mathrm{e}^{-q} \qquad -\mathrm{i}\mathrm{e}^{q} M^{(1)}(x, \rho) = M^{(1)}(x, \rho)\rho^{+}. \tag{65}$$

These properties of the $M^{(1)}$-operator imply the following conditions on the R-matrix:

$$\tau^{+} R^{(11)}(x) = R^{(11)}(x)\rho^{+} \qquad \rho R^{(11)}(x) = R^{(11)}(x)\tau \tag{66}$$

which could be satisfied if $R^{(11)}(x)$ has the following form:

$$R^{(11)}(x) = P_{\rho\tau}\, g(x, \rho^{+}\tau) \tag{67}$$

where $P_{\rho\tau}$ denotes the operator of permutation of $\rho\tau$ variables. Substituting (67) into relation (64) we obtain the equation for the function g:

$$g(x - y, \rho^{+}\tau)M^{(1)}(x, \rho)M^{(1)}(y, \tau) = M^{(1)}(y, \tau)M^{(1)}(x, \rho)g(x - y, \rho^{+}\tau). \tag{68}$$

Making use of the explicit form of the $M^{(1)}$-operator and the formal power-series expansion for the function g with respect to its second argument we can solve this equation and find the function g:

$$g(x, \rho^{+}\tau) = (1 + \rho^{+}\tau)^{-\mathrm{i}x} \tag{69}$$

and therefore

$$R^{(11)}(x) = P_{\rho\tau}(1 + \rho^{+}\tau)^{-\mathrm{i}x}. \tag{70}$$

As this R-matrix intertwines two similar objects, it should satisfy the Yang–Baxter equation (and it really does), but we shall not investigate this issue further.

The last relation which we need to discuss is the intertwining of two $M^{(2)}$-operators:

$$R^{(22)}(x-y)M^{(2)}(x,\rho)M^{(2)}(y,\tau) = M^{(2)}(y,\tau)M^{(2)}(x,\rho)R^{(22)}(x-y). \quad (71)$$

The $M^{(2)}$-operators also satisfy the relations analogous to (65):

$$\rho M^{(2)}(x,\rho) = -\mathrm{i}\mathrm{e}^{-q}M^{(2)}(x,\rho) \qquad M^{(2)}(x,\rho)\mathrm{i}\mathrm{e}^{q} = M^{(2)}(x,\rho)\rho^{+} \quad (72)$$

from where we obtain the analogue of (66):

$$\tau^{+}R^{(22)}(x) = R^{(22)}(x)\rho^{+} \qquad \rho^{+}R^{(22)}(x) = R^{(22)}(x)\tau^{+} \quad (73)$$

and therefore $R^{(22)}$ has the following form:

$$R^{(22)}(x) = P_{\rho\tau}h(x,\tau^{+}\rho). \quad (74)$$

Furthermore, acting as above we find that the unknown function h does coincide with the function g, resulting in the following $R^{(22)}$-matrix:

$$R^{(22)}(x) = P_{\rho\tau}(1+\tau^{+}\rho)^{-\mathrm{i}x}. \quad (75)$$

Now we have completed the derivation of all the needed intertwining relations. The main corollary of these relations is the mutual commutativity of the transfer matrix and both Q-operators:

$$[t(x), Q^{(i)}(y)] = 0 \qquad [Q^{(i)}(x), Q^{(j)}(y)] = 0 \qquad i(j) = 1, 2. \quad (76)$$

5. Wronskian-type functional relations

It was first pointed out in [2] that the Baxter equation (1) which defines the Q-operator could be viewed as the finite-difference analogue of the second-order differential equation which admits two independent solutions. The linear independence of the solutions could be established through the calculation of the Wronskian corresponding to the equation. In the previous section we have constructed two solutions of the Baxter equation and now our task is to prove its linear independence, i.e. to derive the finite-difference analogue of the Wronskian. To solve this problem let us consider in detail the representation of the product (63) of two different Q-operators. In the notation of the previous section the product of two M-operators which enters into the right-hand side of (63) has the following form:

$$\begin{aligned} M_k^{(12)}(x,y,\alpha,\bar\beta,\gamma,\bar\delta) &= M_k^{(1)}(x,\alpha,\bar\beta)M_k^{(2)}(y,\gamma,\bar\delta) \\ &= \mathrm{e}^{-\pi(x+y)/2}\mathrm{e}^{-\mathrm{i}\bar\beta\mathrm{e}^{q_k}}\frac{1}{\Gamma(-\mathrm{i}(x-p_k)+1)}\mathrm{e}^{\mathrm{i}(\alpha-\gamma)\mathrm{e}^{-q_k}}\mathrm{e}^{\pi(y-p_k)}\Gamma(-\mathrm{i}(y-p_k))\mathrm{e}^{\mathrm{i}\bar\delta\mathrm{e}^{q_k}}. \end{aligned} \quad (77)$$

Changing the holomorphic variables according to (58) we obtain

$$\begin{aligned} M_k^{(12)}(x,y,\xi_1,\xi_2,\bar\xi_1',\bar\xi_2') &= \mathrm{e}^{-\pi(x+y)/2}\mathrm{e}^{-\mathrm{i}/\sqrt{2}(\bar\xi_1'+\bar\xi_2')\mathrm{e}^{q_k}} \\ &\quad\times\frac{1}{\Gamma(-\mathrm{i}(x-p_k)+1)}\mathrm{e}^{\mathrm{i}\sqrt{2}\xi_2\mathrm{e}^{-q_k}}\mathrm{e}^{\pi(y-p_k)}\Gamma(-\mathrm{i}(y-p_k))\,\mathrm{e}^{\mathrm{i}/\sqrt{2}(\bar\xi_1'-\bar\xi_2')\mathrm{e}^{q_k}}. \end{aligned} \quad (78)$$

This equation demonstrates that the kernel of $M^{(1)}(x)M^{(2)}(y)$ does not depend on the variable ξ_1 and for calculation of $Q^{(1)}(x)Q^{(2)}(y)$ the dependence of (78) on the variable $\bar\xi_1'$ is irrelevant because the integration over ξ', $\bar\xi'$ in (63) results in deleting $\bar\xi_1'$ from (78). In such a way for the calculation of $Q^{(1)}(x)Q^{(2)}(y)$ we can use instead of $M_k^{(12)}(x, y, \xi_1, \xi_2, \bar{\xi}_1', \bar{\xi}_2')$ the following reduced object:

$$\tilde{M}_k^{(12)}(x, y, \xi, \bar{\xi}') = \mathrm{e}^{-\pi(x+y)/2}\mathrm{e}^{-\mathrm{i}/\sqrt{2}\bar{\xi}'\mathrm{e}^{q_k}} \times \frac{1}{\Gamma(-\mathrm{i}(x-p_k)+1)}\mathrm{e}^{\mathrm{i}\sqrt{2}\xi\mathrm{e}^{-q_k}}\mathrm{e}^{\pi(y-p_k)}\Gamma(-\mathrm{i}(y-p_k))\,\mathrm{e}^{-\mathrm{i}/\sqrt{2}\bar{\xi}'\mathrm{e}^{q_k}}. \tag{79}$$

Note that $\tilde{M}^{(12)}(x, y)$ is nothing else but the kernel of $M^{(1)}(x)M^{(2)}(y)$ on the space $\pi\left(\Gamma^{(1)} \times \Gamma^{(2)}\right)$. Now let us expand the exponents which contain $\xi, \bar{\xi}$ on the right-hand side of (79) and move all the factors depending on p_k to the right:

$$\tilde{M}_k^{(12)}(x, y, \xi, \bar{\xi}') = \mathrm{e}^{-\pi(x+y)}\sum_{n,m=0}\frac{(\mathrm{i}\sqrt{2}\xi)^n}{n!}(-\mathrm{i}\bar{\xi}'/\sqrt{2})^m\mathrm{e}^{(m-n)q_k} \times\sum_{l=0}^{m}\frac{(-1)^{m-l}}{l!(m-l)!}\frac{\Gamma(-\mathrm{i}(y-p_k)-m+l)}{\Gamma(-\mathrm{i}(x-p_k)+1+n-m+l)}\mathrm{e}^{\pi(y-p_k)}. \tag{80}$$

The summation over l in (80) gives

$$\sum_{l=0}^{m}\frac{(-1)^{m-l}}{l!(m-l)!}\frac{\Gamma(-\mathrm{i}(y-p_k)-m+l)}{\Gamma(-\mathrm{i}(x-p_k)+1+n-m+l)} = \frac{(-1)^m}{m!}\frac{\Gamma(-\mathrm{i}(y-p_k)-m)}{\Gamma(-\mathrm{i}(x-p_k)+n+1)}\frac{\Gamma(-\mathrm{i}(x-y)+m+n+1)}{\Gamma(-\mathrm{i}(x-y)+n+1)} \tag{81}$$

and we arrive at the following expression for $\tilde{M}_k^{(12)}(x, y, \xi, \bar{\xi}')$:

$$\tilde{M}_k^{(12)}(x, y, \xi, \bar{\xi}') = \mathrm{e}^{-\pi(x+y)}\sum_{n,m=0}\frac{(\mathrm{i}\sqrt{2}\xi)^n}{n!}\frac{(\mathrm{i}\bar{\xi}'/\sqrt{2})^m}{m!}\mathrm{e}^{(m-n)q_k} \times\frac{\Gamma(-\mathrm{i}(y-p_k)-m)}{\Gamma(-\mathrm{i}(x-p_k)+n+1)}\frac{\Gamma(-\mathrm{i}(x-y)+m+n+1)}{\Gamma(-\mathrm{i}(x-y)+n+1)}\mathrm{e}^{\pi(y-p_k)}. \tag{82}$$

Now let x and y be

$$x = z_+ = z + \mathrm{i}(l+\tfrac{1}{2}) \qquad y = z_- = z - \mathrm{i}(l+\tfrac{1}{2}) \tag{83}$$

where l is an integer (half-integer). For these values of spectral parameters (82) takes the following form:

$$\tilde{M}_k^{(12)}(z_+, z_-, \xi, \bar{\xi}') = \mathrm{e}^{-\pi z}\sum_{n,m=0}\frac{(\mathrm{i}\sqrt{2}\xi)^n}{n!}\frac{(\mathrm{i}\bar{\xi}'/\sqrt{2})^m}{m!}\mathrm{e}^{(m-n)q_k} \times\frac{\Gamma(-\mathrm{i}(z_- - p_k)-m)}{\Gamma(-\mathrm{i}(z_+ - p_k)+n+1)}\frac{\Gamma(2l+m+n+2)}{\Gamma(2l+n+2)}\mathrm{e}^{\pi(z_- - p_k)}. \tag{84}$$

Furthermore, we need to consider (82) for the opposite shift of spectral parameters

$$x = z_- - \mathrm{i}\epsilon \qquad y = z_+ + \mathrm{i}\epsilon. \tag{85}$$

We have introduced infinitesimal ϵ in (85) to remove an ambiguity which arises in (82) for these x and y:

$$\tilde{M}_k^{(12)}(z_-, z_+, \xi, \bar{\xi}') = \mathrm{e}^{-\pi z}\sum_{n,m=0}\frac{(\mathrm{i}\sqrt{2}\xi)^n}{n!}\frac{(\mathrm{i}\bar{\xi}'/\sqrt{2})^m}{m!}\mathrm{e}^{(m-n)q_k} \times\frac{\Gamma(-\mathrm{i}(z_+ - p_k)-m)}{\Gamma(-\mathrm{i}(z_- - p_k)+n+l)}\frac{\Gamma(-2l-2\epsilon+m+n)}{\Gamma(-2l-2\epsilon+n)}\mathrm{e}^{\pi(z_+ - p_k)}. \tag{86}$$

For $\epsilon \to 0$ the fraction of Γ-functions in (86) takes the following values:

$$\lim_{\epsilon\to 0}\frac{\Gamma(-2l-2\epsilon+m+n)}{\Gamma(-2l-2\epsilon+n)} = \begin{cases} \dfrac{\Gamma(-2l+m+n)}{\Gamma(-2l+n)} & n, m \geqslant 2l+1 \\ (-1)^m \dfrac{(2l-n)!}{(2l-n-m)!} & 2l \geqslant n+m \geqslant 0 \\ \dfrac{\Gamma(n+m-2l)}{\Gamma(n-2l)} & n \geqslant 2l \geqslant m \\ 0 & \text{otherwise.} \end{cases} \tag{87}$$

Apparently, the vanishing of (87) in the fourth region manifests the triangularity of the operator $\tilde{M}_k^{(12)}(z_-, z_+)$, therefore for the calculation of the trace of the product over k of these operators we need to consider only the part of (87), which corresponds to the first two regions. Thus, the resulting expression for the twice-reduced operator has the following form:

$$\tilde{\tilde{M}}_k^{(12)}(z_-, z_+, \xi, \bar{\xi}') = A_k(z, l, \xi, \bar{\xi}') + B_k(z, l, \xi, \bar{\xi}') \tag{88}$$

where A contains the part of the right-hand side of (86) with the summation over n, m in the region $n, m \geqslant 2l+1$, B contains the summation over n, m in the region $2l \geqslant n+m \geqslant 0$. In other words, the degrees of $\xi, \bar{\xi}'$ in A and B have no intersection and therefore for the calculation of the product $Q^{(1)}(z_-)Q^{(2)}(z_+)$ these two parts will multiply coherently:

$$\begin{aligned} Q^{(1)}(z_-)Q^{(2)}(z_+) &= \int \prod_{k=1}^{N} \mathrm{d}\mu(\xi_k) \tilde{\tilde{M}}_N^{(12)}(z_-, z_+, \xi_1, \bar{\xi}_N) \\ &\quad \times \tilde{\tilde{M}}_{N-1}^{(12)}(z_-, z_+, \xi_N, \bar{\xi}_{N-1}) \cdots \tilde{\tilde{M}}_1^{(12)}(z_-, z_+, \xi_2, \bar{\xi}_1') \\ &= \int \prod_{k=1}^{N} \mathrm{d}\mu(\xi_k)\, A_N(z, l, \xi_1, \bar{\xi}_N) A_{N-1}(z, l, \xi_N, \bar{\xi}_{N-1}) \cdots A_1(z, l, \xi_2, \bar{\xi}_1) \\ &\quad + \int \prod_{k=1}^{N} \mathrm{d}\mu(\xi_k)\, B_N(z, l, \xi_1, \bar{\xi}_N) B_{N-1}(z, l, \xi_N, \bar{\xi}_{N-1}) \cdots B_1(z, l, \xi_2, \bar{\xi}_1). \end{aligned} \tag{89}$$

Let us consider first A. For convenience we will shift the values of n, m by $2l+1$, then

$$\begin{aligned} A_k(z, l, \xi, \bar{\xi}) &= (\xi\bar{\xi}')^{2l+1} \mathrm{e}^{-\pi z} \sum_{n,m=0} \frac{(\mathrm{i}\sqrt{2}\xi)^n}{n!} \frac{(\mathrm{i}\bar{\xi}'/\sqrt{2})^m}{(m+2l+1)!} \mathrm{e}^{(m-n)q_k} \\ &\quad \times \frac{\Gamma(-\mathrm{i}(z_- - p_k) - m)}{\Gamma(-\mathrm{i}(z_+ - p_k) + n + l)} \frac{\Gamma(2l+m+n+2)}{\Gamma(2l+n+2)} \mathrm{e}^{\pi(z_- - p_k)}. \end{aligned} \tag{90}$$

Comparing (90) with (84), we see that they differ from each other by the factor $(\xi\bar{\xi}')^{2l+1}$ and the shift of the factorial $m!$. This difference may be presented as an appropriate transformation of $\tilde{M}_k^{(12)}(z_+, z_-, \xi, \bar{\xi}')$:

$$A_k(z, l, \xi, \bar{\xi}') = \int \mathrm{d}\mu(\zeta)\, \mathrm{d}\mu(\zeta')\, g_l(\xi, \bar{\zeta}) \tilde{M}_k^{(12)}(z_+, z_-, \zeta, \bar{\zeta}') f_l(\zeta', \bar{\xi}') \tag{91}$$

where

$$g_l(\xi, \bar{\zeta}) = (\xi)^{2l+1} \mathrm{e}^{\xi\bar{\zeta}} \qquad f_l(\zeta, \bar{\xi}) = (\bar{\xi})^{2l+1} \sum_{n=0} \frac{(\zeta\bar{\xi})^n}{(n+2l+1)!}. \tag{92}$$

These two functions possess the following property:

$$\int \mathrm{d}\mu(\xi) f_l(\zeta, \bar{\xi}) g_l(\xi, \bar{\zeta}') = \mathrm{e}^{\zeta\bar{\zeta}'}. \tag{93}$$

The right-hand side of (93) is the δ-function in the holomorphic representation. However, note that

$$\int \mathrm{d}\mu(\xi) g_l(\zeta, \bar{\xi}) f_l(\xi, \bar{\zeta}') = \sum_{n=2l+1} \frac{(\zeta\bar{\zeta}')^n}{n!}. \tag{94}$$

Taking into account (93), we immediately obtain

$$\int \prod_{k=1}^{N} \mathrm{d}\mu(\xi_k)\, A_N(z, l, \xi_1, \bar{\xi}_N) A_{N-1}(z, l, \xi_N, \bar{\xi}_{N-1}) \cdots A_1(z, l, \xi_2, \bar{\xi}_1) = Q^{(1)}(z_+) Q^{(2)}(z_-). \tag{95}$$

Our next step is the consideration of the B part of $M^{(12)}(z_-, z_+)$. First of all we shall remove the $\sqrt{2}$ from its holomorphic arguments, because in the integral (89) these factors will cancelled out. Therefore, we need to consider the following expression for B:

$$B_k(z, l, \xi, \bar{\xi}') = \mathrm{e}^{-\pi z} \sum_{t=0}^{2l} \sum_{m=0}^{t} \frac{\xi^{t-m}}{(t-m)!} \frac{\bar{\xi}'^m}{m!} (-1)^m \mathrm{i}^{t+2l+1} \mathrm{e}^{(2m-t)q_k} \times \frac{(2l+m-t)!}{(2l-t)!} \frac{\Gamma(-\mathrm{i}(z-p_k)+l-m+\frac{1}{2})}{\Gamma(-\mathrm{i}(z-p_k)-l+t-m+l/2)} \mathrm{e}^{\pi(z-p_k)}. \tag{96}$$

We intend to compare this operator with the Lax operator $L_k^l(x)$ of the Toda chain with the auxiliary space corresponding to the spin l. As follows from the results of section 2, $L_k^l(x)$ could be obtained by the reduction of the operator $m_k(x)$ defined in (29) to the subspace corresponding to spin l. In the holomorphic representation the kernel of $L_k^l(x)$ could be easily found using the projection:

$$L_k^l(x, \alpha, \bar{\beta}) = m_k(x - \mathrm{i}(l+\tfrac{1}{2})) \frac{(\alpha\bar{\beta})^{2l}}{\Gamma(2l+1)}. \tag{97}$$

(Note that here we again use two-component variables $\alpha_i, \beta_i, i = 1, 2$.) In (97) the operator $m_k(x)$ should be understood as the differential operator, acting on the projection kernel $\frac{(\alpha\bar{\beta})^{2l}}{\Gamma(2l+1)}$. For the calculation of the right-hand side of (97) recall that the operator exponential function in (29) is well defined because

$$[\mathrm{i}(p-x)+l_3, \rho_1^+\rho_2 \mathrm{e}^q] = [\mathrm{i}(p-x)+l_3, \rho_2^+\rho_1 \mathrm{e}^{-q}] = 0 \tag{98}$$

therefore we can expand the exponential function into a formal series and find the action of each term on the projection kernel:

$$m_k(x - \mathrm{i}(l+\tfrac{1}{2})) \frac{(\alpha\bar{\beta})^{2l}}{\Gamma(2l+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\mathrm{i}(p_k-x)+\rho_1^+\rho_1 - l + \frac{1}{2})}{\Gamma(\mathrm{i}(p_k-x)+\rho_1^+\rho_1 - l - n + \frac{1}{2})} \frac{(\mathrm{i}\rho_1^+\rho_2 \mathrm{e}^{q_k})^n}{n!} \times \frac{(\alpha_1\bar{\beta}_2 \mathrm{e}^{q_k} - \alpha_2\bar{\beta}_1 \mathrm{e}^{-q_k})^{2l}}{\Gamma(2l+1)}. \tag{99}$$

Apparently, only $2l$ terms in (99) will survive because the differential operator $(\rho_2)^n$ acts on the polynomial. The result has the following form:

$$L_k^l(x, \alpha, \bar{\beta}) = \sum_{t=0}^{2l} \sum_{m=0}^{t} \mathrm{e}^{(2m-t)q_k} \frac{\Gamma(-\mathrm{i}(x-p_k)-m+l+\frac{1}{2})}{\Gamma(-\mathrm{i}(x-p_k)-m+t-l+\frac{1}{2})} \times (-1)^m \mathrm{i}^{2l+t} \frac{\alpha_1^{2l-t+m} \alpha_2^{t-m} \bar{\beta}_1^{2l-m} \bar{\beta}_2^m}{(2l-t)!(t-m)!m!}. \tag{100}$$

This L-operator defines the transfer matrix of the Toda chain with an auxiliary space and corresponding spin l:

$$t^l(x) = \int \prod_{k=1}^{N} \mathrm{d}^2\mu(\alpha_k) L^l_N(x, \alpha_1, \bar{\alpha}_N) L^l_{N-1}(x, \alpha_N, \bar{\alpha}_{N-1}) \cdots L^l_1(x, \alpha_2, \bar{\alpha}_1). \tag{101}$$

If in this formula we perform the integration over one pair of holomorphic variables, corresponding, for example, to $\alpha_1, \bar{\beta}_1$ in (100), the integrand will still be presented in a factorized form, but with a new, reduced kernel of the L-operator:

$$\begin{aligned}\tilde{L}^l_k(x, \alpha, \bar{\beta}) = \sum_{t=0}^{2l} \sum_{m=0}^{t} \mathrm{e}^{(2m-t)q_k} \frac{\Gamma(-\mathrm{i}(x - p_k) - m + l + \frac{1}{2})}{\Gamma(-\mathrm{i}(x - p_k) - m + t - l + \frac{1}{2})} \\ \times (-1)^m \mathrm{i}^{2l+t} \frac{\alpha_2^{t-m} \bar{\beta}_2^m}{(2l - t)!(t - m)!m!} (2l - t + m)!.\end{aligned} \tag{102}$$

Comparing (102) with (96) we find that

$$B_k(z, l, \xi, \bar{\xi}') = \tilde{L}^l_k(z, \xi, \bar{\xi})\, \mathrm{i}\, \mathrm{e}^{-\pi p_k}. \tag{103}$$

Therefore,

$$\int \prod_{k=1}^{N} \mathrm{d}\mu\, (\xi_k) B_N(z, l, \xi_1, \bar{\xi}_N) B_{N-1}(z, l, \xi_N, \bar{\xi}_{N-1}) \cdots B_1(z, l, \xi_2, \bar{\xi}_1) = \mathrm{i}^N t^l(x) \mathrm{e}^{-\pi P} \tag{104}$$

where

$$P = \sum_{k=0}^{N} p_k \tag{105}$$

is the integral of motion, which commutes with $t^l(x)$. In the derivation of (104) we have moved all the factors $\mathrm{e}^{-\pi p_k}$ to the right to form $\mathrm{e}^{-\pi P}$. Gathering together (89), (95) and (104) we obtain the following functional relations:

$$Q^{(1)}(z - \mathrm{i}(l + \tfrac{1}{2})) Q^{(2)}(z + \mathrm{i}(l + \tfrac{1}{2})) - Q^{(1)}(z + \mathrm{i}(l + \tfrac{1}{2})) Q^{(2)}(z - \mathrm{i}(l + \tfrac{1}{2})) = \mathrm{i}^N t^l(x) \mathrm{e}^{-\pi P}. \tag{106}$$

For $l = 0$ the transfer matrix turns into 1 and we have the simplest Wronskian relation:

$$Q^{(1)}(z - \tfrac{1}{2}\mathrm{i}) Q^{(2)}(z + \tfrac{1}{2}\mathrm{i}) - Q^{(1)}(z + \tfrac{1}{2}\mathrm{i}) Q^{(2)}(z - \tfrac{1}{2}\mathrm{i}) = \mathrm{i}^N \mathrm{e}^{-\pi P}. \tag{107}$$

For the illustration of this identity the reader can use the $Q^{(i)}$-operators for one degree of freedom (47) and (48). In this simplest case (107) reduces to the well known identity for Bessel functions:

$$I_\nu(z) I_{-\nu+1}(z) - I_{-\nu}(z) I_{\nu-1}(z) = -\frac{2\sin(\pi\nu)}{\pi z}. \tag{108}$$

The general case (106) for one degree on freedom is related to Lommel polynomials [8].

The functional relations of the type (106) were first established for a certain field-theoretical model in [2]. In the recent paper of the author with Stroganov [9] we have discussed the analogous relation for the eigenvalues of Q-operators in the case of an isotropic Heisenberg spin chain. Originally, since the Baxter paper [1] the existence of one Q-operator was considered as important alternative for the Bethe ansatz. The relations (106) show the importance of the second Q-operator, which together with the first one give rise to numerous fusion relations (see, e.g., [2, 9]).

6. Discussion

The approach we have considered in the present paper could also be applied to the other with the rational R-matrix, the discrete self-trapping (DST) model, considered in [3]. The quantum determinant of the Lax operator for this model is not unity and the Baxter equation has the following form:

$$t(x)Q(x) = (x - \tfrac{1}{2}\mathrm{i})^N Q(x - \mathrm{i}) + Q(x + \mathrm{i}). \tag{109}$$

The general properties of the Q-operators for the DST model are similar to that of the Toda system. The eigenvalues of one Q-operator are polynomial in the spectrum parameter, while the eigenvalues of the second one are meromorphic functions. In the case of the Toda system the eigenvalues of $Q^{(1)}(x)$ are entire functions, the eigenvalues of $Q^{(2)}(x)$ are meromorphic. For the DST model there also exist functional relations similar to (106).

The most interesting would be the application of the formalism to the case of the XXX-spin chain. The situation here is the following. In [4] we have constructed the family of $Q(x, l)$-operators similar to (31). Moreover, from the results of [9] it follows that for an XXX-spin chain there exist basic Q-operators. Making use of the formalism of section 3, it is possible to the find the $M_k^{(i)}(x)$-operators for this case, but the trace of monodromies corresponding to $M_k^{(i)}(x)$ diverges. This puzzle deserves further investigation.

Another interesting point we want to discuss is the relation of our $M_k^{(i)}(x)$-operators with the quantum linear problem for the Lax operator (5). In the classical case the linear problem is the main ingredient of the inverse scattering method, at the same time for the quantum theory it seems to be unnecessary (see, for example, the excellent review on the subject [10]). However, let us consider the following problem:

$$\psi_{n+1}(x) = L_n(x)\psi_n(x) \tag{110}$$

where $L_n(x)$ is given in (5) and ψ_n is a two-component quantum operator. From the multiplication rules (43) we obtain

$$(L_n(x))_{ij} M_n^{(1)}(x) \begin{pmatrix} 1 \\ \rho \end{pmatrix}_j = \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i M_n^{(1)}(x - \mathrm{i}). \tag{111}$$

Now let us define the operator

$$\left(\psi_n^{(1)}\right)_i(x) = \mathrm{Tr}\left(\prod_{k=n}^{N} M_k^{(1)}(x) \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i \prod_{k=1}^{n-1} M_k^{(1)}(x - \mathrm{i})\right) \tag{112}$$

where the trace is taken over the auxiliary space. Apparently, due to (111) the operator (112) does satisfy equation (110). For $n = 1$, the solution has the following form:

$$\left(\psi_1^{(1)}\right)_i(x) = \mathrm{Tr}\left(\prod_{k=1}^{N} M_k^{(1)}(x) \begin{pmatrix} 1 \\ \rho \end{pmatrix}_i\right) = Q^{(1)}(x) \begin{pmatrix} 1 \\ \mathrm{i}\mathrm{e}^{q_N} \end{pmatrix}_i \tag{113}$$

where in the last step we have used the explicit form of the $Q^{(1)}(x)$-operator for the calculation of the trace. On the other hand, the solution (113) translated to the period N by the monodromy (8), due to (111) is given by

$$\left(\psi_{N+1}^{(1)}\right)_i(x) = \mathrm{Tr}\left(\begin{pmatrix} 1 \\ \rho \end{pmatrix}_i \prod_{k=1}^{N} M_k^{(1)}(x - \mathrm{i})\right) = Q^{(1)}(x - \mathrm{i}) \begin{pmatrix} 1 \\ \mathrm{i}\mathrm{e}^{q_N} \end{pmatrix}_i. \tag{114}$$

In other words, we obtain

$$T_{ij}(x)\left(\psi_1^{(1)}\right)_j(x) = \frac{Q^{(1)}(x - \mathrm{i})}{Q^{(1)}(x)}\left(\psi_1^{(1)}\right)_i(x). \tag{115}$$

This equation may be understood as the quantum analogue of the property of Bloch solutions, which are eigenvectors of the translation to the period.

Similarly, we can consider the second solution. Indeed, from the multiplications rules (43) for the $M_n^{(2)}(x)$ we obtain

$$(L_n(x))_{ij} M_n^{(2)}(x) \begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_j = \begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_i M_n^{(2)}(x - \mathrm{i}). \tag{116}$$

Therefore, the operator

$$\left(\psi_n^{(2)}\right)_i(x) = \mathrm{Tr}\left(\prod_{k=n}^{N} M_k^{(2)}(x) \begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_i \prod_{k=1}^{n-1} M_k^{(2)}(x - \mathrm{i})\right) \tag{117}$$

possesses the same properties as (112). The initial value of (117) is given by

$$\left(\psi_1^{(2)}\right)_i(x) = \mathrm{Tr}\left(\prod_{k=1}^{N} M_k^{(2)}(x) \begin{pmatrix} -\rho^+ \\ 1 \end{pmatrix}_i\right) = Q^{(2)}(x) \begin{pmatrix} -\mathrm{i}\mathrm{e}^{q_1} \\ 1 \end{pmatrix}_i \tag{118}$$

where again on the last step we have used the explicit form of $M^{(2)}(x)$. As above we obtain

$$T_{ij}(x)\left(\psi_1^{(2)}\right)_j(x) = \frac{Q^{(2)}(x - \mathrm{i})}{Q^{(2)}(x)}\left(\psi_1^{(2)}\right)_i(x). \tag{119}$$

In such a way using $M_n^{(i)}(x)$-operators we succeeded in the construction of the operators which may be interpreted as the quantum analogues of the Bloch functions of the corresponding linear problem. In the classical theory of finite-zone 'potentials', two Bloch solutions of the linear problem, as functions of the spectral parameter are actually projections of the Backer–Akhiezer function, which is the single-valued meromorphic function on a hyper-elliptic surface. In the quantum case the Bloch functions (112) and (117) do not possess the branching points (in a weak sense) which is the trace of the projection in the classical case, therefore their intimate relation is somehow hidden and it will be very interesting to uncover this relationship.

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